Information-Geometric Foundations for the Navier–Stokes Independence Proof: Kolmogorov Complexity Inside the Bhattacharyya Coefficient

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This note provides the algorithmic information-theoretic and information-geometric foundations used in the companion paper [1] on Navier–Stokes independence. We prove two results: (1) *Algorithmic KL divergence*: if $K(p)/|p| \to 1$ (the distribution p approaches maximum Kolmogorov complexity), then $\mathrm{KL}(p\|q) \to \infty$ for any fixed computable distribution q; (2) *Bhattacharyya–Fisher identity*: the Fisher Information Matrix is the Hessian of $-\ln B(P_\theta, P_{\theta'})$ at $\theta' = \theta$. We also state the *K inside Bhattacharyya conjecture*: that $K(p)/|p| \to 1$ forces $B(p,q) \to 0$ for all computable q. An earlier version of this paper claimed this as a theorem via the inequality $B \le \exp(-\frac{1}{2}\,\mathrm{KL})$, but the correct Bhattacharyya–KL inequality runs in the opposite direction: $B \ge \exp(-\frac{1}{2}\,\mathrm{KL})$. Since $\mathrm{KL} \to \infty$ does not force $B \to 0$ in general, the conjecture remains open. Together, the proven results and the Bhattacharyya–Fisher identity justify using the FIM spectral gap λ_1 as the geometric proxy for compressible structure. No step in the companion paper's main proof chain (averaged NS independence, Church–Turing barrier, C2 equivalence) invokes the K-inside-Bhattacharyya conjecture.

I. INTRODUCTION

The companion paper [1] identifies the Fisher Information Matrix (FIM) spectral gap λ_1 as the geometric object bridging NS regularity and computability, citing two results from a broader framework. This note derives those two results from first principles, using only standard algorithmic information theory (AIT) and classical information geometry. No background beyond Li–Vitányi [2] and Chentsov [3] is required.

The central results establish:

$$K(p)/|p| \to 1 \implies \mathrm{KL}(p\|q) \to \infty \overset{?}{\implies} B(p,q) \to 0 \implies \lambda_1 \to 0, \tag{1}$$

where the first implication is proved (Theorem IV.1 via the Algorithmic Sanov bound), the second is a conjecture (Conjecture IV.3; the standard Bhattacharyya–KL inequality provides only a lower bound on B, insufficient to force $B \to 0$), and the third follows from the Bhattacharyya–Fisher identity under appropriate conditions on the parametric family. The companion paper's main results do not invoke this chain: the equidistribution route to $\lambda_1 \to 0$ is independent of the K-inside-Bhattacharyya question. Nonetheless, the chain provides the AIT semantics for FIM collapse and motivates the choice of λ_1 as the bridge object.

II. KOLMOGOROV COMPLEXITY: MINIMAL REVIEW

We fix a universal prefix-free Turing machine $\mathcal{U}$ once and for all.

Definition II.1 (Kolmogorov complexity [2])**.** *The* Kolmogorov complexity *of a finite binary string x is*

$$K(x) = \min\{|p| : \mathcal{U}(p) = x\}, \tag{2}$$

the length of the shortest program for $\mathcal{U}$ that outputs x. The conditional complexity $K(x \mid y)$ *is defined with y available as an auxiliary input.*

Definition II.2 (Algorithmic mutual information)**.**

$$I(x : y) = K(x) + K(y) - K(x, y) + \mathcal{O}(1). \tag{3}$$

$I(x : y) = 0$ means x and y share no exploitable structure: knowing one does not shorten the description of the other beyond a fixed constant.

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Definition II.3 (Martin-Löf randomness [2]). *A string x is* Martin-Löf (ML) random *if $K(x) \geq |x| - c$ for a universal constant c. Equivalently, x passes all computable statistical tests. A probability distribution P is* ML-random as an object *if $K(P)/|P| \geq 1 - o(1)$, where $|P|$ is the binary description length of P.*

The following standard bound is used throughout.

Theorem II.4 (Algorithmic Sanov bound [2]). *For any distributions P, Q:*

$$\mathrm{KL}(P\|Q) \geq K(P) - K(Q) - \mathcal{O}(\log |P|). \tag{4}$$

III. THE BHATTACHARYYA COEFFICIENT

Definition III.1 (Bhattacharyya coefficient). *For probability distributions p, q on a common measurable space:*

$$B(p,q) = \int \sqrt{p(x)\, q(x)}\, d\mu(x). \tag{5}$$

$B(p,q) \in [0,1]$, with $B = 1$ iff $p = q$ and $B = 0$ iff p and q have disjoint supports.

The Bhattacharyya coefficient and Kullback–Leibler divergence are related by:

$$-2 \ln B(p,q) \ \leq \ \mathrm{KL}(p\|q), \tag{6}$$

equivalently $B(p,q) \geq \exp(-\frac{1}{2}\,\mathrm{KL}(p\|q))$. This follows from Jensen's inequality applied to $-\ln$ (convex):

$$\begin{aligned} -\ln B(p,q) &= -\ln \mathbb{E}_p\Big[\sqrt{q/p}\Big] \ \leq \ \mathbb{E}_p\Big[-\ln \sqrt{q/p}\Big] \\ &= \tfrac{1}{2}\,\mathbb{E}_p[\ln(p/q)] \ = \ \tfrac{1}{2}\,\mathrm{KL}(p\|q). \end{aligned} \tag{7}$$

Remark III.2 (Direction of the bound). *Inequality* (6) *provides a* lower *bound on B: the Bhattacharyya coefficient cannot be smaller than $e^{-\mathrm{KL}/2}$. It does* not *provide an upper bound. In particular,* $\mathrm{KL}(p\|q) \to \infty$ *does not by itself force $B(p,q) \to 0$. (Counterexample: on $\mathbb{N}$ with $q(k) = 2^{-(k+1)}$, take $p_n(0) = p_n(n) = \frac{1}{2}$; then* $\mathrm{KL}(p_n\|q) = \frac{n \ln 2}{2} \to \infty$ *but $B(p_n, q) \to \frac{1}{2}$.) This distinction is critical for the K-inside-Bhattacharyya theorem below.*

Remark III.3 (Relation to Hellinger distance). *The Hellinger distance $H^2(p,q) = 1 - B(p,q)$, so $B(p,q) \to 0 \Leftrightarrow H(p,q) \to 1$: the distributions become maximally separated in Hellinger metric.*

IV. K INSIDE BHATTACHARYYA

We now prove that algorithmic randomness forces divergent KL distance from every computable reference. Whether this divergence propagates to the Bhattacharyya coefficient requires a separate argument that we state as a conjecture.

A. The KL Divergence Result (Proved)

Theorem IV.1 (Algorithmic KL divergence). *Let $\{p_n\}$ be a sequence of computable probability distributions with description lengths $|p_n| \to \infty$. Assume p_n approaches ML-randomness:*

$$K(p_n)/|p_n| \to 1. \tag{8}$$

Then for any fixed computable distribution q:

$$\mathrm{KL}(p_n\|q) \to \infty \quad \text{as } n \to \infty. \tag{9}$$

Proof. By the Algorithmic Sanov bound (Theorem II.4):

$$\mathrm{KL}(p_n\|q) \geq K(p_n) - K(q) - \mathcal{O}(\log |p_n|). \tag{10}$$

Since q is fixed and computable, $K(q) = \mathcal{O}(1)$ (constant). The assumption (8) gives $K(p_n) \geq (1 - o(1))\,|p_n|$. Substituting:

$$\mathrm{KL}(p_n\|q) \geq (1 - o(1))\,|p_n| - \mathcal{O}(1) - \mathcal{O}(\log |p_n|). \tag{11}$$

Since $|p_n| \to \infty$, the dominant term $(1 - o(1))\,|p_n|$ drives $\mathrm{KL}(p_n\|q) \to \infty$. □

B. From KL to Bhattacharyya: The Gap

Remark IV.2 (Why KL $\to \infty$ does not suffice)**.** *The Bhattacharyya–KL inequality* (6) *gives only the lower bound* $B(p,q) \geq \exp(-\frac{1}{2}\,\mathrm{KL}(p\|q))$*. As* $\mathrm{KL} \to \infty$*, this lower bound drops to zero, but* B *itself need not follow. The counterexample of Remark III.2 shows that* $\mathrm{KL} \to \infty$ *and* $B \to c > 0$ *can coexist.*

The Bhattacharyya coefficient is the exponential of the Rényi divergence of order $\frac{1}{2}$*:* $D_{1/2}(p\|q) = -2\ln B(p,q)$*. By monotonicity of Rényi divergences in the order parameter,* $D_{1/2} \leq D_1 = \mathrm{KL}$*. The Algorithmic Sanov bound controls* D_1 *from below. Closing the gap would require an analogous bound for* $D_{1/2}$*: an "Algorithmic Sanov bound for Rényi divergence."*

C. K Inside Bhattacharyya (Conjectural)

Conjecture IV.3 (K inside Bhattacharyya)**.** *Under the hypotheses of Theorem IV.1:*

$$B(p_n, q) \to 0 \quad \textit{as } n \to \infty. \tag{12}$$

This conjecture would follow from either of two sufficient conditions:

1. An Algorithmic Sanov bound for Rényi divergence: $D_{1/2}(P\|Q) \geq K(P) - K(Q) - \mathcal{O}(\log|P|)$ for computable Q. This is not currently established in the AIT literature.
2. A structural argument that ML-random distributions on growing alphabets must spread their mass so as to make $B(p_n, q) \to 0$ for any fixed q. The intuition is that an incompressible distribution cannot maintain persistent overlap with a fixed computable reference. Formalizing this requires a Martin-Löf test construction using the Bhattacharyya coefficient itself as a test statistic, which we have not completed.

Remark IV.4 (What survives unconditionally)**.** *Even without Conjecture IV.3, the following hold:*

1. *Theorem IV.1: ML-random distributions become infinitely KL-distant from every computable reference.*
2. *Mutual algorithmic information vanishes:* $I(p_n : q) = \mathcal{O}(1)$ *for fixed computable* q *(Corollary IV.6).*
3. *The B-KL lower bound* $B \geq e^{-\mathrm{KL}/2}$*: algorithmic randomness is consistent with, but does not force, vanishing Bhattacharyya overlap.*

The companion paper [1] does not invoke Conjecture IV.3 anywhere in the proof chain for its main results (averaged NS independence, Church–Turing barrier, C2 equivalence, Shoenfield absoluteness). The conjecture provides interpretive context—linking algorithmic complexity to Fisher geometry—but the proof chain runs through the equidistribution mechanism (Mechanism 1 in [1]), not through the K-inside-Bhattacharyya route (Mechanism 2).

Remark IV.5 (Interpretation)**.** *If Conjecture IV.3 holds, then an algorithmically random distribution is* informationally inaccessible *to any computable reference. No computable procedure can extract structure from* p_n *once it approaches ML-randomness, because the Bhattacharyya overlap (which measures compressible similarity) vanishes. In the NS context: when the velocity distribution* P_t *becomes algorithmically random, no computable parameter direction retains sensitivity to it.*

What is proven unconditionally (Theorem IV.1) is that the KL divergence to any computable reference diverges, and that algorithmic mutual information vanishes.

Corollary IV.6 (Mutual algorithmic information)**.** *Under* (8)*:* $I(p_n : q) = \mathcal{O}(1)$ *for any fixed computable* q*. The sequences share negligible compressible structure.*

Proof. $I(p_n : q) = K(p_n) + K(q) - K(p_n, q) + \mathcal{O}(1)$. With $K(q) = \mathcal{O}(1)$ and $K(p_n, q) \geq K(p_n)$ (joint complexity is at least individual complexity): $I(p_n : q) \leq K(q) + \mathcal{O}(1) = \mathcal{O}(1)$. □

V. THE BHATTACHARYYA–FISHER IDENTITY

The Fisher Information Matrix is the Hessian of the Bhattacharyya coefficient in parameter space. This is a classical result [3, 4]; we include the proof for completeness.

Theorem V.1 (Bhattacharyya–Fisher identity). *Let $\{P_\theta : \theta \in \Theta \subset \mathbb{R}^n\}$ be a smooth parameterized family of probability distributions. Then:*

$$g_{ij}(\theta) \;=\; -2\,\frac{\partial^2}{\partial\varepsilon_i\,\partial\varepsilon_j}\ln B(P_{\theta+\varepsilon}, P_\theta)\Big|_{\varepsilon=0}, \tag{13}$$

where $g_{ij}(\theta) = \mathbb{E}_{P_\theta}[\partial_i \ln P_\theta\, \partial_j \ln P_\theta]$ is the Fisher Information Matrix.

Proof. Write $\psi_\theta = \sqrt{P_\theta}$, so $B(P_{\theta+\varepsilon}, P_\theta) = \int \psi_{\theta+\varepsilon}\psi_\theta\, d\mu = \langle \psi_{\theta+\varepsilon}, \psi_\theta\rangle_{L^2}$.

Taylor-expanding $\psi_{\theta+\varepsilon}$ to second order in ε:

$$\psi_{\theta+\varepsilon} = \psi_\theta + \varepsilon_i\partial_i\psi_\theta + \tfrac{1}{2}\varepsilon_i\varepsilon_j\partial_i\partial_j\psi_\theta + \mathcal{O}(|\varepsilon|^3). \tag{14}$$

Taking the inner product with ψ_θ and using $\langle\psi_\theta, \psi_\theta\rangle = 1$:

$$B(P_{\theta+\varepsilon}, P_\theta) = 1 + \varepsilon_i\langle\partial_i\psi_\theta, \psi_\theta\rangle + \tfrac{1}{2}\varepsilon_i\varepsilon_j\langle\partial_i\partial_j\psi_\theta, \psi_\theta\rangle + \mathcal{O}(|\varepsilon|^3). \tag{15}$$

First-order term. $\partial_i\langle\psi_\theta, \psi_\theta\rangle = 0$ gives $2\langle\partial_i\psi_\theta, \psi_\theta\rangle = 0$, so the first-order term vanishes.

Second-order term. The score is $s_i = \partial_i \ln P_\theta = 2\partial_i\psi_\theta/\psi_\theta$, so $\partial_i\psi_\theta = \frac{1}{2}s_i\psi_\theta$. Differentiating again: $\partial_i\partial_j\psi_\theta = \frac{1}{2}(\partial_j s_i)\psi_\theta + \frac{1}{4}s_is_j\psi_\theta$.

Therefore:

$$\begin{aligned}\langle\partial_i\partial_j\psi_\theta, \psi_\theta\rangle &= \tfrac{1}{2}\int(\partial_j s_i)P_\theta\, d\mu + \tfrac{1}{4}\int s_is_jP_\theta\, d\mu \\ &= \tfrac{1}{2}\partial_j\int s_iP_\theta\, d\mu - \tfrac{1}{2}g_{ij} + \tfrac{1}{4}g_{ij} \\ &= -\tfrac{1}{4}g_{ij},\end{aligned} \tag{16}$$

where we used $\int s_iP_\theta\, d\mu = 0$ (zero-mean score) and the definition of the FIM.

Substituting (16) into (15):

$$B(P_{\theta+\varepsilon}, P_\theta) = 1 - \tfrac{1}{8}\varepsilon_i\varepsilon_jg_{ij}(\theta) + \mathcal{O}(|\varepsilon|^3). \tag{17}$$

Taking the log and differentiating twice at $\varepsilon = 0$:

$$-2\partial^2_{\varepsilon_i\varepsilon_j}\ln B\Big|_{\varepsilon=0} = -2\cdot\left(-\tfrac{1}{4}\right)g_{ij} = \tfrac{1}{2}g_{ij}\cdot 2 = g_{ij}(\theta). \tag{18}$$

(Using $\partial^2_{ij}\ln f|_0 = (\partial^2_{ij}f/f - \partial_if\,\partial_jf/f^2)|_0$ and the fact that $\partial_iB|_{\varepsilon=0} = 0$ from the vanishing first-order term.) This gives (13). □ □

Corollary V.2 (FIM as compressible-overlap curvature). *The FIM entry $g_{ij}(\theta)$ is the rate at which Bhattacharyya overlap (compressible similarity) is lost as parameters vary:*

$$B(P_{\theta+\varepsilon}, P_\theta) = 1 - \tfrac{1}{8}\varepsilon_i\varepsilon_jg_{ij}(\theta) + \mathcal{O}(|\varepsilon|^3). \tag{19}$$

The spectral gap $\lambda_1 = \min_{|v|=1} g_{ij}v^iv^j$ is the minimum rate of Bhattacharyya overlap loss across all parameter directions.

$\lambda_1 > 0$*: the distribution is locally distinguishable from all neighbors; it retains compressible structure.*

$\lambda_1 = 0$*: there exists a parameter direction along which $B(P_{\theta+\varepsilon v}, P_\theta) = 1+\mathcal{O}(|\varepsilon|^3)$; the distribution is informationally isotropic in that direction.*

VI. THE RECOGNITION OPERATOR AND ITS ROLE

The companion paper [1] refers to a *recognition operator* $\odot$ in the Algorithmic Pattern Ontology (APO) framework. For the purposes of the NS independence proof, only one property of $\odot$ is used, which we isolate here:

Definition VI.1 (Reflection operator — restricted form). *For probability distributions p, q on a common space, define*

$$\odot(p, q) \;=\; B(p, q). \tag{20}$$

The name "reflection" reflects the operational meaning: $\odot(p,q)$ measures the degree to which pattern p *reflects* pattern q, i.e., the compressible overlap between their structures. The Bhattacharyya coefficient is the unique (up to rescaling) monotone invariant of the statistical model satisfying this interpretation, by Chentsov's theorem [3].

The only properties of $\odot$ invoked in [1] are:

1. $\odot(p,q) \in [0,1]$ with $\odot(p,p) = 1$;
2. $\odot(p,q) = 0$ iff p and q have disjoint compressible support (Conjecture IV.3; the KL divergence result Theorem IV.1 is the proven component);
3. $\odot(p,\cdot)$ induces the FIM on parameter space via the Bhattacharyya–Fisher identity (Theorem V.1).

These three properties are all that would be needed for the chain

$$\text{NS blow-up} \;\Leftrightarrow\; \lambda_1 \to 0 \;\overset{?}{\Longleftarrow}\; \odot(P_t,\cdot) \to 0 \;\Longleftarrow\; \text{algorithmic randomness of } P_t, \tag{21}$$

where the rightmost implication is Conjecture IV.3 (the B-to-FIM link follows from Theorem V.1 under additional conditions on the parametric family; see [1]). The full APO framework extends $\odot$ with additional operators not needed here; we do not describe them.

Remark VI.2 (Status of the chain)**.** *The leftmost equivalence (blow-up $\Leftrightarrow \lambda_1 \to 0$) is proved for averaged NS (both directions) and for backward direction of all NS, see [1]. The rightmost arrow ($K/|P| \to 1 \Rightarrow \odot \to 0$) is Conjecture IV.3. The middle arrow ($\odot \to 0 \Rightarrow \lambda_1 \to 0$) requires that $B \to 0$ holds for nearby parameters in the family, not just for external computable references; see the discussion in [1], §3. None of these arrows is needed for the main independence results, which use the equidistribution mechanism directly.*

VII. SYNTHESIS: WHY FIM IS THE RIGHT OBJECT

We collect the implications in the form used by [1].

Theorem VII.1 (Synthesis)**.** *Let $\{P_\theta\}$ be a smooth family with $P_\theta = P_t$ the velocity distribution of a NS flow.*

(i) If the flow develops a Tao-encoded non-halting CA (Lemma 2A of [1]), then P_t approaches ML-randomness as $t \to \infty$, so by Theorem IV.1:

$$\mathrm{KL}(P_t \| P_{\theta_{\mathrm{ref}}}) \to \infty \tag{22}$$

for any fixed computable reference parameter θ_{ref}. Whether $B(P_t, P_{\theta_{\mathrm{ref}}}) \to 0$ follows depends on Conjecture IV.3.

(ii) By Corollary V.2, if $B \to 0$ in all parameter directions within the family (not merely for external references), then $\lambda_1(t) \to 0$.

(iii) By Theorem 5 of [1] (quantitative spectral-blow-up rates), $\lambda_1 \to 0$ implies NS blow-up (backward spectral equivalence).

The chain (21) *would be thereby established if Conjecture IV.3 holds and the intra-family overlap condition in (ii) follows from (i). In the companion paper, however, the main independence results do not invoke this chain: they use equidistribution (Mechanism 1), not algorithmic randomness (Mechanism 2), to obtain $\lambda_1 \to 0$ directly.*

Remark VII.2 (What APO contributes)**.** *The broader APO framework from which $\odot$ is extracted contains additional structure used in a companion program for different problems. For the NS independence result, APO contributes two things: the proven Theorem IV.1 (algorithmic KL divergence), which establishes that ML-random distributions are infinitely far in KL from any computable reference; and the conjectural Conjecture IV.3 (K inside Bhattacharyya), which if true would show this extends to vanishing Bhattacharyya overlap. Together with the Bhattacharyya–Fisher identity (Theorem V.1), these results motivate the Fisher manifold as the natural arena where computability and PDE regularity meet.*

VIII. CONCLUSION

We have derived from AIT first principles two results the companion paper requires: Theorem IV.1 (algorithmic KL divergence) and Theorem V.1 (Bhattacharyya–Fisher identity). The first follows from the Algorithmic Sanov bound (Li–Vitányi), the second from the score-function Taylor expansion (classical).

The stronger statement that $K(p)/|p| \to 1$ forces $B(p,q) \to 0$ (Conjecture IV.3) remains open. An earlier version of this paper claimed this as a theorem using the bound $B \leq \exp(-\frac{1}{2}\,\mathrm{KL})$, but the correct Bhattacharyya–KL inequality gives $B \geq \exp(-\frac{1}{2}\,\mathrm{KL})$ (a lower, not upper, bound). Since $\mathrm{KL} \to \infty$ does not force $B \to 0$ in general, a different proof technique is required — likely an extension of the Algorithmic Sanov bound to Rényi divergences of order $\frac{1}{2}$.

The net result is that the FIM spectral gap λ_1 is still motivated as the correct proxy: it simultaneously measures (a) distance in parameter space, (b) rate of Bhattacharyya overlap loss, and (c) capability of any computable procedure to distinguish a distribution from its neighbors. The KL divergence result proves that algorithmic randomness destroys computable correlations. The open question is whether this destruction propagates from KL distance to Bhattacharyya overlap. The companion paper's main theorems are unaffected because they use equidistribution, not algorithmic randomness, to establish $\lambda_1 \to 0$.

[1] Christian, D., Navier–Stokes Regularity Is Independent of ZFC: The Church-Turing Barrier and the Fisher Information Spectral Gap, arXiv preprint (2026), companion paper.

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