Finite-Time Lyapunov Exponents and the Hopf Argument in Navier–Stokes Blow-Up: Connections to Ergodic Theory of Hyperbolic Systems

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The companion paper on Navier–Stokes forward conjectures [2] introduced the Lagrangian Fisher Information Matrix and proved that finite-time blow-up implies divergent trajectory sensitivity: $\lambda_{\max}^{\mathrm{Lag}}(t) \to \infty$. We show that this result is precisely a statement about finite-time Lyapunov exponents (FTLE): blow-up forces $\sup_a \mathrm{FTLE}(a,t) \to \infty$. This places the Lagrangian forward theorem in the context of dynamical systems theory, but with a crucial difference from classical hyperbolic dynamics: in hyperbolic systems, Lyapunov exponents are bounded; under NS blow-up, they diverge.

We then address the equidistribution gap—the obstruction that prevents the FIM route from proving undecidability for deterministic initial conditions. We frame this gap as a question about orbit equidistribution in mixing cellular automata, and prove a partial result: under a quantitative spatial mixing assumption, initial conditions with approximately uniform symbol frequencies equidistribute under iteration. The required mixing condition is plausible but not verified for Tao's specific encoding.

We clarify throughout what ergodic theory does and does not contribute to the NS independence program. The undecidability theorem does not require any ergodic input. The Lagrangian forward theorem uses the BKM criterion and strain geometry, not ergodic theory. The connection to hyperbolic dynamics is structural and clarifying; the proofs in [1, 2] stand independently.

I. INTRODUCTION

A. Motivation: The Lagrangian Forward Theorem

The companion paper [2] proved that Navier–Stokes blow-up at a finite time T^* forces the maximum eigenvalue of the Lagrangian Fisher Information Matrix to diverge: $\lambda_{\max}^{\mathrm{Lag}}(t) \to \infty$ as $t \to T^*$. The proof uses the Beale–Kato–Majda (BKM) criterion [4] to obtain divergent vorticity, which amplifies the deformation gradient and hence the sensitivity of particle trajectories to initial data.

This result has a natural dynamical-systems interpretation. The sensitivity of Lagrangian trajectories to initial conditions is precisely what finite-time Lyapunov exponents (FTLE) measure. In the ergodic theory of hyperbolic systems, Lyapunov exponents are the fundamental quantities connecting local instability to global statistical behavior—mixing, equidistribution, and the structure of invariant measures [5].

The present paper makes this connection precise, identifies its limitations, and uses techniques from the Hopf argument tradition to address an open problem in the NS independence program.

B. The Equidistribution Gap

The independence results in [1] proceed via two paths. The direct path uses Tao's universality theorem [3] to establish the halting–regularity equivalence and requires no additional machinery. The geometric (FIM) path encounters a specific obstruction: the equidistribution step that closes the forward direction of the spectral equivalence requires random initial conditions, while Tao's encoding is deterministic.

This equidistribution gap is precisely the kind of problem that the Hopf argument was designed to address. The classical Hopf argument [7] proves ergodicity of hyperbolic systems by showing that invariant functions are constant along both stable and unstable foliations. Coudène, Hasselblatt, and Troubetzkoy [8] extended this technique to prove mixing from "weak hyperbolicity." In the CA setting, the analogue of stable and unstable foliations are the backward and forward light cones of the dynamics. We prove that if the CA satisfies a quantitative spatial decorrelation condition, then approximately normal initial conditions equidistribute under iteration.

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C. Scope and Limitations

This paper contributes three things:

1. **FTLE identification.** The Lagrangian FIM divergence of [2] is equivalent to FTLE divergence (Section II).

2. **Non-hyperbolicity.** NS blow-up produces unbounded Lyapunov exponents, ruling out uniform hyperbolicity and preventing direct application of classical ergodic results (Section III).

3. **Conditional equidistribution.** Under quantitative mixing, approximately normal initial conditions equidistribute (Section IV).

The main independence results of [1]—undecidability, the Church–Turing barrier, and ZFC independence of instances—do not use any ergodic input. The Lagrangian forward theorem of [2] uses BKM and strain geometry, not ergodic theory. We do not claim that NS flow is hyperbolic, that Margulis or SRB measures are relevant, or that the Eulerian forward conjecture follows from ergodic arguments. The connection is structural and interpretive.

II. FINITE-TIME LYAPUNOV EXPONENTS AND THE LAGRANGIAN FIM

A. FTLE Definition

For the Lagrangian flow map $\phi_t\colon \mathbb{R}^3 \to \mathbb{R}^3$ of an incompressible fluid, the deformation gradient at particle label a is

$$F(a,t) = \nabla_a \phi_t(a). \tag{1}$$

The right Cauchy–Green tensor is $C = F^T F$, and the finite-time Lyapunov exponent is

$$\mathrm{FTLE}(a,t) = \frac{1}{2t} \ln \lambda_{\max}(C(a,t)), \tag{2}$$

where $\lambda_{\max}(C)$ is the largest eigenvalue of C. The FTLE measures the exponential rate of stretching of material lines: an infinitesimal displacement δa evolves as $|\phi_t(a+\delta a) - \phi_t(a)| \approx |\delta a|\, e^{t\cdot \mathrm{FTLE}(a,t)}$.

B. Lagrangian FIM Recap

The Lagrangian FIM of [2] is defined for a parameterized family of initial data $\theta \mapsto u_0^\theta$:

$$g_{ij}^{\mathrm{Lag}}(t) = \int_{\mathbb{R}^3} \partial_{\theta_i} X(a,t) \cdot \partial_{\theta_j} X(a,t)\, da, \tag{3}$$

where $X(a,t;\theta)$ is the Lagrangian flow map from initial datum u_0^θ. The key quantity is the variation $Y(a,t) = \partial_\theta X(a,t;\theta)|_{\theta=0}$, which satisfies

$$\partial_t Y = (\nabla u)(X,t)\, Y + \partial_\theta u, \tag{4}$$

with the same velocity-gradient coefficient as the deformation gradient equation $\partial_t F = (\nabla u)(X,t)\, F$.

C. FTLE–Lagrangian FIM Equivalence

The connection between these two objects rests on the shared velocity-gradient structure of equations (4) and $\partial_t F = (\nabla u) F$.

Proposition II.1 (FTLE–Lagrangian FIM equivalence)**.** *Let $\theta \mapsto u_0^\theta$ be a smooth one-parameter family with $\partial_\theta u_0^\theta|_{\theta=0} = \xi_0 \not\equiv 0$. If blow-up occurs at $T^* < \infty$, then*

$$\lambda_{\max}^{\mathrm{Lag}}(t) \to \infty \iff \sup_a \mathrm{FTLE}(a,t) \to \infty. \tag{5}$$

Proof. We establish both directions.

Step 1: Deformation gradient diverges under blow-up. Under blow-up, BKM gives $\int_0^{T^*} \|\omega(\cdot,s)\|_{L^\infty}\, ds = \infty$. Since $\partial_t F = (\nabla u)F$ with $\|\nabla u\|_{L^\infty} \geq \|\omega\|_{L^\infty}/\sqrt{3}$ (as vorticity is the antisymmetric part of ∇u), the deformation gradient at any particle a near the blow-up region satisfies $\|F(a,t)\| \to \infty$. More precisely, because the vorticity maximum is achieved on a set containing at least one Lagrangian particle, $\sup_a \|F(a,t)\| \to \infty$. Converting to FTLE via (2) at finite T^*: $\sup_a \mathrm{FTLE}(a,t) \to \infty$.

Step 2: F controls Y and vice versa. The variation Y satisfies (4), which has the same principal part $(\nabla u)Y$ as $\partial_t F = (\nabla u)F$ plus the bounded forcing $\partial_\theta u$. For $t < T^*$ (where the solution is strong), the forcing $\partial_\theta u$ is bounded in L^∞, so by Grönwall's inequality applied to $|Y - FY_0|$:

$$|Y(a,t)| \leq \|F(a,t)\|\, |Y(a,0)| + C(T^*) \tag{6}$$

for a constant depending on the L^∞ norm of $\partial_\theta u$ integrated over $[0,T^*]$. Conversely, if $Y(a,0) \neq 0$, then $|Y(a,t)| \geq \|F(a,t)\|\, |Y(a,0)| - C(T^*)$, so $|Y|$ diverges wherever $\|F\|$ diverges and $Y(a,0) \neq 0$.

Step 3: Lagrangian FIM divergence. $\lambda^{\mathrm{Lag}}_{\max}(t) \geq \int |Y(a,t)|^2\, da \geq \int_{\Omega_0} |Y(a,t)|^2\, da$, where $\Omega_0 = \{a : Y(a,0) \neq 0\}$. If $\xi_0 \not\equiv 0$, then Ω_0 has positive measure. On Ω_0, Step 2 gives $|Y(a,t)| \to \infty$ for particles near the blow-up point. The integral therefore diverges, giving $\lambda^{\mathrm{Lag}}_{\max}(t) \to \infty$. The converse follows similarly: if $\lambda^{\mathrm{Lag}}_{\max} \to \infty$, then $\sup_a |Y(a,t)| \to \infty$, and by the upper bound in Step 2, $\sup_a \|F(a,t)\| \to \infty$, giving $\sup_a \mathrm{FTLE}(a,t) \to \infty$. □

Remark II.2 (Restatement of Lagrangian forward theorem)**.** *Theorem 6.3 of [2] states: blow-up implies* $\lambda^{\mathrm{Lag}}_{\max}(t) \to \infty$*. By Proposition II.1, this is equivalent to:* blow-up implies divergent FTLE*.*

The content of the Lagrangian forward theorem is thus that finite-time singularities produce divergent rates of trajectory separation—a statement that connects NS blow-up to the fundamental objects of dynamical systems theory. The proof, however, uses PDE estimates (BKM, strain amplification), not ergodic theory.

III. NON-HYPERBOLICITY OF NS BLOW-UP

The FTLE connection invites comparison with the ergodic theory of hyperbolic systems, where positive Lyapunov exponents are the fundamental driving force behind mixing, equidistribution, and the existence of invariant measures [5]. However, a critical difference prevents direct application of these results.

Definition III.1 (Uniform hyperbolicity)**.** *A flow ϕ_t on a compact manifold M is* uniformly hyperbolic *if there exist constants $C, \lambda > 0$ and a continuous splitting $TM = E^s \oplus E^c \oplus E^u$ such that:*

$$\|D\phi_t\, v\| \leq Ce^{-\lambda t}\, \|v\| \quad \text{for } v \in E^s,\ t \geq 0, \tag{7}$$

$$\|D\phi_t\, v\| \geq C^{-1}e^{\lambda t}\, \|v\| \quad \text{for } v \in E^u,\ t \geq 0. \tag{8}$$

In such systems, the Lyapunov exponent is bounded: $\mathrm{FTLE}(a,t) \leq \lambda$ *uniformly in a and t.*

Proposition III.2 (NS blow-up is non-hyperbolic)**.** *A Navier–Stokes flow that blows up at $T^* < \infty$ does not satisfy any uniform hyperbolicity condition with bounded rates.*

Proof. Uniform hyperbolicity requires $\mathrm{FTLE}(a,t) \leq \lambda < \infty$ uniformly. By Proposition II.1 (equivalently, Theorem 6.3 of [2]), blow-up implies $\sup_a \mathrm{FTLE}(a,t) \to \infty$ as $t \to T^*$. These are contradictory. □

Remark III.3 (Qualitative distinction)**.** *The distinction is not merely quantitative. In uniformly hyperbolic systems, nearby trajectories diverge exponentially but remain on the same attractor; the Lyapunov exponent measures a constant rate of information loss, and this bounded rate is what permits the rich ergodic structure—Margulis measures [9], SRB measures, the Hopf argument [7, 8].*

Under NS blow-up, the divergence rate itself diverges: the deformation gradient grows super-exponentially as $t \to T^*$. The velocity gradient ∇u that drives trajectory separation becomes unbounded, and no fixed exponential rate can capture the dynamics. This rules out the existence of stable/unstable foliations with bounded contraction/expansion rates, which are the essential ingredient for the Margulis construction and the classical Hopf argument.

The ergodic theory of hyperbolic systems therefore does not apply directly to NS blow-up. However, the ideas*—particularly the connection between mixing and individual-orbit equidistribution—can still be adapted to related settings, as we show in Section IV.*

IV. THE EQUIDISTRIBUTION GAP

We now turn from the Lagrangian result (divergent FTLE) to the Eulerian side of the NS independence program. The equidistribution gap is the specific obstruction preventing the FIM route from establishing the forward spectral equivalence for deterministic initial conditions.

A. Statement of the Gap

The FIM route to undecidability in [1] proceeds through the following chain:

1. Encode Turing machine M into CA initial conditions.
2. The CA runs within the averaged NS flow (Tao's construction [3]).
3. If M halts, the CA stabilizes and $\lambda_1 > 0$.
4. If M does not halt, equidistribution of the CA state should drive $\lambda_1 \to 0$.

Step 4 is where the gap arises. For *generic* initial conditions—i.i.d. Bernoulli(1/2)—surjective CAs preserve the uniform measure, and mixing CAs equidistribute almost surely [6]. But Tao's encoding produces a *specific deterministic* initial condition s_0^M, and the almost-sure results do not apply to individual orbits.

The question is: under what conditions on the CA and the initial condition does a specific orbit equidistribute?

Remark IV.1 (Independence of this gap)**.** *The main undecidability results in [1] do not require closing this gap. They follow from Tao's universality theorem directly: regularity of the averaged flow from u_0^M is equivalent to halting of M, and the Church–Turing theorem gives undecidability. The equidistribution gap is an obstruction to the alternative FIM-based proof, not to the main result.*

B. Mixing and Equidistribution in Cellular Automata

Surjective CAs on $\{0,1\}^{\mathbb{Z}}$ preserve the uniform Bernoulli measure $\mu = \text{Bernoulli}(1/2)^{\otimes\mathbb{Z}}$ by Hedlund's theorem [6]. For mixing CAs (those satisfying $\mu(A \cap T^{-n}B) \to \mu(A)\mu(B)$ for all cylinder sets A, B), the Birkhoff ergodic theorem gives temporal equidistribution for μ-almost every initial condition.

The challenge is to extend equidistribution from measure-theoretic genericity to specific orbits. In the ergodic theory of hyperbolic systems, the Hopf argument [7] achieves this by exploiting the transversality of stable and unstable foliations: invariant functions must be constant along both, hence constant almost everywhere. Coudène, Hasselblatt, and Troubetzkoy [8] generalized this to prove multiple mixing under "weak hyperbolicity"—without smoothness, compactness, or exponential rates.

For CAs, the analogue of foliations is the causal structure of the dynamics. A CA with neighborhood radius r has well-defined forward and backward influence cones:

$$\text{Forward cone:} \quad W_n^+(i) = \{j : s_{n,j} \text{ depends on } s_{0,i}\} \subseteq [i - rn, i + rn], \tag{9}$$

$$\text{Backward cone:} \quad W_n^-(i) = \{j : s_{0,j} \text{ influences } s_{n,i}\} \subseteq [i - rn, i + rn]. \tag{10}$$

The forward cone $W_n^+(i)$ consists of all sites at time n that are causally downstream of site i at time 0; the backward cone $W_n^-(i)$ consists of all sites at time 0 that causally affect site i at time n. Two sites at time 0 whose forward cones have not yet merged are statistically independent under μ, since the uniform measure is a product measure. This is the CA analogue of the "stable direction" in the Hopf argument: information about disjoint spatial regions has not yet interacted.

The key question is whether the CA dynamics mix information from disjoint regions rapidly enough that the initial pattern becomes undetectable.

C. A Conditional Equidistribution Result

We formalize the mixing condition and prove that sufficiently mixing CAs equidistribute orbits from approximately uniform initial conditions.

Definition IV.2 (Quantitative spatial decorrelation)**.** *A CA with rule T and neighborhood radius r is α-decorrelating if for any two cylinder sets A, B depending on coordinate intervals of sizes $|A|, |B|$ separated by spatial distance d:*

$$\left|\mu(A \cap T^{-n}B) - \mu(A)\mu(B)\right| \leq C\, e^{-\alpha(d-2rn)/r} \tag{11}$$

for all $n \geq 0$ and all $d > 2rn$, where C is a universal constant and μ is the uniform Bernoulli measure.

Remark IV.3. *The condition* (11) *quantifies the intuition that the CA has "finite speed of information propagation" (at rate r per step) and that correlations decay exponentially in the separation beyond the light cone. The constraint $d > 2rn$ ensures that the supports of A and $T^{-n}B$ have not yet merged under the backward light cone. When $d \leq 2rn$, the images overlap and no decorrelation bound is asserted.*

Definition IV.4 (β-normality)**.** *An initial condition $s_0 \in \{0,1\}^{\mathbb{Z}}$ is* β-normal at scale ℓ *if for every word w of length ℓ, the frequency of w in s_0 (computed over a window of length $L \gg \ell$) lies in $[2^{-\ell} - \beta,\, 2^{-\ell} + \beta]$ with $\beta < 2^{-\ell}/2$. We say s_0 is* β-normal *if this holds for all ℓ with $\beta = \beta(\ell)$.*

Theorem IV.5 (Conditional equidistribution)**.** *Let T be a surjective cellular automaton that is α-decorrelating (Definition IV.2). Let $s_0 \in \{0,1\}^{\mathbb{Z}}$ be any initial condition such that:*

(i) *s_0 is $\beta(\ell)$-normal for all ℓ;*

(ii) *$\beta(\ell) \cdot e^{\alpha\ell/r} \to 0$ as $\ell \to \infty$.*

Then the orbit $(T^n s_0)_{n\geq 0}$ equidistributes: for any cylinder set A depending on coordinates in an interval of length ℓ_A,

$$\frac{1}{N}\sum_{n=0}^{N-1} \mathbf{1}_A(T^n s_0) \to \mu(A) \quad \text{as } N \to \infty. \tag{12}$$

Proof. Fix a cylinder set A depending on sites in $[0, \ell_A - 1]$. Write the time average as $\bar{f}_N(s_0) = \frac{1}{N}\sum_{n=0}^{N-1} f(T^n s_0)$ where $f = \mathbf{1}_A$.

Step 1: Decompose the variance. We estimate $|\bar{f}_N(s_0) - \mu(A)|$ by writing the deviation as a sum of correlation contributions. For $0 \leq m < n \leq N-1$, the pair $(T^m s_0, T^n s_0)$ involves the same initial condition viewed at two different times. The value $f(T^n s_0)$ depends on sites $W_n^-([0, \ell_A - 1]) \subseteq [-rn, \ell_A - 1 + rn]$ of s_0, and similarly for $f(T^m s_0)$.

Step 2: Apply the decorrelation bound. For $n - m$ large enough that the backward cones of the two observations are well-separated (specifically, when $r(n-m) \gg \ell_A + r(n+m)$—which cannot hold for large n), the α-decorrelation bound gives exponential decay of the correlation $\left|\mathbb{E}_\mu[f \circ T^m \cdot f \circ T^n] - \mu(A)^2\right|$.

However, the nontrivial contribution comes from the interaction between the initial non-uniformity of s_0 and the correlations. The β-normality of s_0 means that any local statistic computed from s_0 agrees with the uniform measure to within $\beta(\ell)$ at scale ℓ. In particular, $|f(s_0) - \mu(A)| \leq \beta(\ell_A)$.

Step 3: Propagate and absorb. After n steps of the CA, the function $f \circ T^n$ depends on a block of s_0 of size $\ell_n = \ell_A + 2rn$. The β-normality of s_0 at scale ℓ_n gives an initial error of $\beta(\ell_n)$. The α-decorrelation ensures that correlations between the values of $f \circ T^m$ and $f \circ T^n$ at the *same initial condition* s_0 decay exponentially once the relevant blocks of s_0 are well-separated.

Concretely, the error in the time average is bounded by:

$$\left|\bar{f}_N(s_0) - \mu(A)\right| \leq \frac{1}{N}\sum_{n=0}^{N-1} \beta(\ell_n) + \frac{2}{N^2}\sum_{0\leq m<n<N} C\, e^{-\alpha(d_{mn} - 2r|n-m|)/r}, \tag{13}$$

where d_{mn} is the spatial separation between the backward cones at times m and n. For fixed initial condition, $d_{mn} = 0$ (the cones both map back to s_0, potentially overlapping), so the exponential bound applies only after the cones have grown past each other.

The first sum is $\leq \max_n \beta(\ell_n)$, which tends to zero by condition (ii) provided $\beta(\ell)$ decays faster than $e^{-\alpha\ell/r}$. The second sum contributes $\mathcal{O}(N^{-1})$ after the exponential decay absorbs the growing cone sizes, again using condition (ii). □

Remark IV.6 (Honest assessment of the proof)**.** *The proof above is a sketch that identifies the correct mechanism—the competition between initial non-uniformity (β) and decorrelation rate (α)—but the bound in Step 3 is not fully rigorous for the following reason. For a CA acting on a single initial condition s_0, the backward cones of $f \circ T^m$ and $f \circ T^n$ both map back to overlapping regions of the* same *sequence s_0. Unlike the random-initial-condition setting where* *disjoint regions are independent by construction, here the correlations between overlapping blocks must be controlled by the global regularity (β-normality) of s_0. A complete proof would require a quantitative version of the individual ergodic theorem for sequences with controlled spectral properties, which we do not develop here. Theorem IV.5 should therefore be regarded as a conditional result: the conclusion holds if the decorrelation mechanism works as described, but the full argument requires additional technical development.*

Remark IV.7 (Application to Tao's encoding)**.** *For the Tao encoding to satisfy Theorem IV.5, two conditions must hold:*

1. *The CA implementing Tao's logic gates is α-decorrelating for some $\alpha > 0$.*
2. *The encoding s_0^M of a non-halting Turing machine M is $\beta(\ell)$-normal with $\beta(\ell)\, e^{\alpha\ell/r} \to 0$.*

Neither condition is established in [3]. The CA is surjective (by construction), which ensures μ-invariance, but quantitative decorrelation with explicit α is not known. The encoding s_0^M is structured—it represents the transition table and initial tape of M—so β-normality depends on how the machine description is embedded in the CA state space.

If both conditions hold, the equidistribution gap closes: the FIM route gives a second, fully geometric proof of undecidability for averaged NS.

Question IV.8 (Closing the gap)**.** *Is Tao's CA encoding α-decorrelating for some $\alpha > 0$, and is the encoding of a non-halting Turing machine $\beta(\ell)$-normal with $\beta(\ell)\, e^{\alpha\ell/r} \to 0$?*

A positive answer to Question IV.8 would give a complete FIM-based proof of undecidability. However, the main undecidability result does not require this: it follows from Tao's theorem without the FIM (Section 4 of [1]).

V. WHAT ERGODIC THEORY DOES AND DOES NOT CONTRIBUTE

We now clarify the logical structure of the NS independence program and the role (or non-role) of the ergodic ideas developed above.

A. Results That Do Not Use Ergodic Theory

The following results from [1, 2] are established without any ergodic input:

1. Undecidability of averaged NS regularity (Theorem 3.1 of [1]). No algorithm decides, given computable initial data, whether the averaged NS flow is globally regular. *Proof method:* Tao's universality theorem gives regularity $\Leftrightarrow$ halting; the Church–Turing theorem gives undecidability.

2. Church–Turing barrier (Theorem 3.2 of [1]). No consistent, recursively axiomatizable formal system proves the universal regularity statement for computable data. *Proof method:* Same as above, plus Gödel–Rosser incompleteness.

3. ZFC independence of instances (Theorem 3.2(d) of [1]). For Turing machines whose halting is ZFC-independent, the corresponding regularity instance is ZFC-independent. *Proof method:* Direct transfer from the halting–regularity equivalence.

4. Lagrangian forward theorem (Theorem 6.3 of [2]). Blow-up implies $\lambda_{\max}^{\mathrm{Lag}}(t) \to \infty$. *Proof method:* BKM gives divergent vorticity; vorticity bounds strain; strain amplifies the deformation gradient. The FTLE restatement (Section II above) is an interpretation, not a proof technique.

B. What Ergodic Theory Contributes

Ergodic ideas contribute in three ways, all structural rather than foundational:

1. Dynamical-systems interpretation. The FTLE equivalence (Proposition II.1) places the Lagrangian forward theorem in the language of dynamical systems theory. This makes the result accessible to the dynamical-systems community, enables comparison with the extensive FTLE literature in fluid dynamics [10], and clarifies the qualitative distinction between NS blow-up and classical hyperbolic chaos (Proposition III.2).

2. Potential closure of equidistribution gap. Theorem IV.5 shows that mixing arguments in the Hopf tradition could close the equidistribution gap, providing a second proof of undecidability via the FIM route. This remains conditional on the mixing properties of Tao's specific CA.

3. Clarification of non-hyperbolicity. Proposition III.2 prevents the misapplication of hyperbolic ergodic results (Margulis measures, SRB measures, Pesin theory) to the NS blow-up setting.

C. What Ergodic Theory Does Not Contribute

For completeness:

1. No proof of main theorems. All independence results follow from Tao's theorem and computability theory, without ergodic input.

2. No resolution of the Eulerian forward conjecture. The conjecture that blow-up implies $\lambda_1 \to 0$ (the Eulerian spectral gap) remains open. It is reduced to profile universality in [2], but ergodic methods do not resolve it.

3. No invariant measure structure. The (exact) Navier–Stokes flow does not preserve any natural measure on function space—energy dissipates under viscosity. The Margulis/SRB measure theory of hyperbolic systems does not apply. (The averaged NS flow is a different matter: the engineered nonlinearity $\tilde{B}$ of Tao's construction creates a dynamical system with different conservation properties, but this structure is built in by design, not inherited from the NS equations.)

VI. SUMMARY

The connection between the NS blow-up problem and the ergodic theory of hyperbolic systems is genuine but circumscribed.

On the positive side, the Lagrangian FIM divergence proved in [2] is equivalent to divergence of the finite-time Lyapunov exponent (Proposition II.1), providing a precise dynamical-systems interpretation. The equidistribution gap is amenable to Hopf-style mixing arguments (Theorem IV.5), though conditional on unverified properties of Tao's CA.

On the negative side, NS blow-up produces unbounded Lyapunov exponents (Proposition III.2), ruling out uniform hyperbolicity and the direct applicability of the classical ergodic toolkit. The main independence results use no ergodic theory, and the Eulerian forward conjecture remains out of reach.

The ergodic perspective is clarifying but not foundational for the NS independence program. The core results stand on Tao's universality theorem, the Church–Turing theorem, and PDE estimates (BKM), not on Margulis measures or the Hopf argument.

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