Scenario I′ and the C2 Boundary: Why the Clay Problem Encodes Its Own Proof-Theoretic Status

D. Christian[1,*]

[1] *The Axioms of Pattern Ontology (APO) Initiative*

(Dated: March 6, 2026)

The main paper [1] establishes a four-way dichotomy (R/F/I/I′) for the exact Navier–Stokes regularity statement ($\star_{\rm ex}$), constrained by the C2 equivalence and Shoenfield absoluteness. The C2 equivalence assigns each scenario a computational character: in F and I, the fluid computes unboundedly and the Church–Turing barrier generates undecidability; in R and I′, computation is bounded and the barrier is absent.

This computational characterization generates a natural hypothesis: the Church–Turing barrier is the *complete* obstruction to provability of ($\star_{\rm ex}$). If true, I′ (regularity true but ZFC-unprovable) does not occur, and the dichotomy reduces to R, F, or I.

We prove three results supporting this hypothesis. First, the *contrapositive chain*: in the no-blow-up branch, the energy barrier bounds computation, persistent FIM distinguishability follows from the backward spectral equivalence, and the K-inside-B contrapositive bounds the Kolmogorov complexity of the velocity distribution (Theorem III.1). Second, *uniform distinguishability*: under ($\star_{\rm ex}$), the BKM integral converges over $[0, \infty)$ and the FIM spectral gap $\lambda_1(t; e)$ is bounded away from zero uniformly in time, by the scale invariance of λ_1 and the finite vorticity budget (Theorem VII.1). Third, a *conditional exclusion*: if regularity proof complexity is bounded by a computable function of the datum, then I′ is excluded (Theorem IV.7).

The hypothesis is not provable from within the C2 framework. We show that proving it in the no-blow-up branch is equivalent to proving ($\star_{\rm ex}$)—the Clay problem itself. The C2 equivalence ensures that every question about the proof-theoretic status of ($\star_{\rm ex}$), including whether I′ holds, reduces to the Clay question. The framework does not resolve the dichotomy; it characterizes each possible resolution and shows that no resolution can be achieved independently of the Clay problem.

I. INTRODUCTION

The Clay Millennium Prize problem asks whether every smooth, finite-energy initial datum on $\mathbb{R}^3$ produces a global smooth Navier–Stokes solution. The main paper [1] proves that for averaged NS, this is false and the decision problem encodes the halting problem. For exact NS, the C2 equivalence (Theorem 8.1 of [1]) establishes:

$$\text{unlimited computation} \iff \text{blow-up exists} \iff \neg(\star_{\rm ex}). \tag{1}$$

Combined with Shoenfield absoluteness [3] (($\star_{\rm ex}$) is Π^1_1, truth value forcing-invariant), this yields four mutually exclusive scenarios [1]:

	ZFC decides	ZFC does not decide
($\star_{\rm ex}$) true	**(R)** Regularity provable	**(I′)** True, unprovable
($\star_{\rm ex}$) false	**(F)** Blow-up provable	**(I)** Blow-up, unprovable

The C2 equivalence assigns each scenario a precise computational character. In scenarios F and I, blow-up exists. Blow-up amplification outpaces viscous dissipation, enabling the exact NS flow to simulate a non-halting Turing machine indefinitely from computable initial data. The system becomes computationally universal: it can encode any Turing machine, and the regularity question for each datum reduces to the halting question for the encoded machine. The Church–Turing barrier engages—no formal system extending PA can uniformly decide regularity, because doing so would decide the halting problem. The mechanism of proof-theoretic difficulty is identified and complete: *unlimited computation.*

In scenarios R and I′, blow-up does not exist. The energy barrier caps each datum at finitely many faithful simulation steps—the fluid computes, but boundedly. No computable datum encodes a non-halting machine indefinitely. The

* daniel@freereason.org

Church–Turing barrier does not engage: there is no halting encoding, no uniform undecidability, no computational obstruction to proof.

This computational characterization generates a natural hypothesis about provability, in two strengths.

Conjecture I.1 (C2 provability hypothesis—weak form)**.** *The Church–Turing barrier is the only obstruction to proving* $(\star_{\rm ex})$ *in ZFC. That is: if* $(\star_{\rm ex})$ *is true (no blow-up, barrier absent), then* $\mathrm{ZFC} \vdash (\star_{\rm ex})$.

The weak form eliminates I′: regularity cannot be true and unprovable. The dichotomy reduces to R, F, or I.

Conjecture I.2 (C2 provability hypothesis—strong form)**.** *The Church–Turing barrier is the* complete *obstruction to ZFC-decidability of* $(\star_{\rm ex})$. *That is: ZFC decides* $(\star_{\rm ex})$ *one way or the other.*

(a) If blow-up exists, $(\star_{\rm ex})$ *is false and* $\mathrm{ZFC} \vdash \neg(\star_{\rm ex})$.

(b) If no blow-up exists, $(\star_{\rm ex})$ *is true and* $\mathrm{ZFC} \vdash (\star_{\rm ex})$.

The strong form eliminates both I′ and I. The dichotomy collapses to two: R or F. The Clay problem is ZFC-decidable.

The two forms differ on the blow-up side. In scenario I, blow-up exists but ZFC cannot prove it. The (c)⇒(b) direction of C2 says: if $(\star_{\rm ex})$ is false, computable blow-up data should exist (by blow-up stability under H^1 perturbation). A computable blow-up datum is a Σ_1 witness: the numerical simulation diverges in finite time, and ZFC can verify this computation. If computable blow-up data exist, ZFC proves $\neg(\star_{\rm ex})$, and scenario I is eliminated. But blow-up stability for 3D NS is not proven (Remark 8.3 of [1]); without it, Shoenfield absoluteness provides a Δ^2_1-definable witness, which is hyperarithmetic but not necessarily computable, and ZFC may not be able to certify it.

Summary of what each form assumes and eliminates:

Form	**Additional assumption**	**Surviving scenarios**
Unconstrained	None	R, F, I, I′
Weak (Conjecture I.1)	None beyond C2	R, F, I
Strong (Conjecture I.2)	Blow-up stability	R, F

Both forms share the same motivation:

- C2 is a biconditional equivalence, not merely an implication. It accounts for the logical structure of $(\star_{\rm ex})$ through a single mechanism—unlimited computation via blow-up amplification—leaving no identified room for a second source of proof-theoretic difficulty.

- In the no-blow-up branch, the flow is informationally tame (Theorem III.1): bounded computation, persistent distinguishability, compressible distributions.

- Under $(\star_{\rm ex})$, the FIM spectral gap is uniformly bounded away from zero (Theorem VII.1) and the BKM integral converges, giving a single finite regularity certificate for each datum.

- All counter-models of $(\star_{\rm ex})$ in the I′ scenario are ill-founded (Shoenfield), a strong model-theoretic constraint.

The strong form additionally requires that blow-up, if it exists, is stable enough to produce a computable witness—a PDE assumption, not a logical one.

Neither conjecture is provable from within the C2 framework. Proving the weak form in the no-blow-up branch requires proving $(\star_{\rm ex})$; proving the strong form in the blow-up branch requires proving blow-up stability. Both are open PDE problems. The purpose of this note is to make these conjectures precise, prove the results that support them, and identify the exact gaps that separate them from theorems. Those gaps, as we show, are the Clay problem itself (for the weak form) and blow-up stability (for the strong form).

II. PRELIMINARIES

We recall the key results from [1] and the APO companion [2].

A. The C2 Equivalence

Theorem II.1 (C2 equivalence, Theorem 8.1 of [1]). *The following are equivalent for exact NS:*

(a) There exists computable $u_0 \in C^\infty \cap L^2$ whose exact NS flow simulates a non-halting Turing machine for all $n \in \mathbb{N}$.

(b) There exists computable $u_0^* \in C^\infty \cap L^2$ whose exact NS solution blows up in finite time.

(c) $(\star_{\rm ex})$ is false.

Proposition II.2 (Energy barrier, Proposition 7.4 of [1]). *The energy inequality limits any finite-energy exact NS datum to at most $B(e) = \lfloor \left\| u_0^{(e)} \right\|_{L^2} /(C_K \nu) \rfloor$ faithful CA simulation steps, where C_K depends on the averaging kernel and e is the computable index.*

The C2 contrapositive gives:

Corollary II.3 (No blow-up implies bounded computation). *If $(\star_{\rm ex})$ is true, then for every computable datum e, the exact NS flow from $u_0^{(e)}$ can simulate at most $B(e) < \infty$ Turing machine steps. No computable datum supports unlimited computation.*

B. Backward Spectral Equivalence

Theorem II.4 (Backward spectral equivalence, Theorem 3.2 of [1]). *For any strong NS solution (averaged or exact):*

$$\lambda_1(t) \to 0 \text{ as } t \to T^* \implies \text{blow-up at } T^*.$$

The proof is via Grönwall on the FIM evolution:

$$\lambda_1(t) \;\geq\; \lambda_1(0) \exp\Big(-C' \int_0^t \|\omega(\cdot, s)\|_{L^\infty} \, ds\Big). \tag{2}$$

If $\lambda_1 \to 0$, the BKM [5] integral diverges, forcing blow-up.

Corollary II.5 (Persistent distinguishability). *If $(\star_{\rm ex})$ is true, then for each computable datum e and each finite $T > 0$:*

$$\lambda_1(t; e) > 0 \quad \text{for all } t \in [0, T]. \tag{3}$$

The velocity distribution $P_t^{(e)}$ remains locally distinguishable from its parametric neighbors at every finite time.

Proof. If $\lambda_1(t^*; e) = 0$ for some finite t^*, then by the Grönwall lower bound (2) with $\lambda_1(0) > 0$, the BKM integral $\int_0^{t^*} \|\omega\|_{L^\infty} = \infty$, giving blow-up at or before t^*. This contradicts $(\star_{\rm ex})$. □

C. K-inside-B and Bhattacharyya–Fisher

Theorem II.6 (K inside Bhattacharyya, Appendix C of [1]). *If $\{p_n\}$ are computable distributions with $|p_n| \to \infty$ and $K(p_n)/|p_n| \to 1$, then $B(p_n, q) \to 0$ for any fixed computable q.*

Theorem II.7 (Bhattacharyya–Fisher identity, Appendix C of [1]). *For a smooth parameterized family $\{P_\theta\}$:*

$$g_{ij}(\theta) \;=\; -2 \frac{\partial^2}{\partial \varepsilon_i \partial \varepsilon_j} \ln B(P_{\theta+\varepsilon}, P_\theta) \Big|_{\varepsilon=0}. \tag{4}$$

Corollary II.8 (FIM as distinguishability curvature). *$\lambda_1 > 0$ if and only if P_θ is locally distinguishable from all its parametric neighbors (Bhattacharyya overlap $B < 1$ to second order in every direction). By Cramér–Rao [6], this is equivalent to the existence of a finite-variance unbiased estimator for the parameter θ.*

III. THE CONTRAPOSITIVE CHAIN

We now assemble the contrapositives of the K-inside-B chain into a single result for the no-blow-up branch.

Theorem III.1 (Informational tameness in the no-blow-up branch)**.** *Assume* ($\star_{\mathrm{ex}}$) *is true. Then for each computable datum e, the exact NS flow satisfies all of the following:*

(i) ***Bounded computation.*** *The flow simulates at most* $B(e) < \infty$ *Turing machine steps (Corollary II.3).*

(ii) ***Persistent distinguishability.*** $\lambda_1(t; e) > 0$ *for all finite t (Corollary II.5).*

(iii) ***Bhattacharyya persistence.*** *For the parameterized family* $\{P_\theta^{(t)}\}$ *centered at datum e, the local Bhattacharyya coefficient satisfies* $B(P_{\theta+\varepsilon}^{(t)}, P_\theta^{(t)}) \geq 1 - \frac{1}{4}\lambda_{\max}(t)\,|\varepsilon|^2 + O(|\varepsilon|^3)$*, bounded away from zero for each t and small* ε *(Theorem II.7).*

(iv) ***Bounded Kolmogorov complexity (contrapositive).*** *The velocity distribution* $P_t^{(e)}$ *does not approach Martin-Löf randomness [4]:*

$$\limsup_{t\to\infty} \frac{K(P_t^{(e)})}{|P_t^{(e)}|} \;<\; 1. \tag{5}$$

Proof. (i) is the C2 contrapositive (Corollary II.3).
(ii) is the backward spectral equivalence contrapositive (Corollary II.5).
(iii) follows from (ii) via the Bhattacharyya–Fisher identity (Theorem II.7): $\lambda_1 > 0$ implies $B < 1$ to second order, hence B is bounded away from zero locally.
(iv) is the contrapositive of Theorem II.6: if $K(P_t)/\,|P_t| \to 1$, then $B(P_t, q) \to 0$ for all computable q. This would force $B(P_t^{(\theta)}, P_t^{(\theta')}) \to 0$ for distinct computable parameters (taking q as the distribution at a neighboring parameter value). By the BF identity, this forces $g_{ij} \to 0$, hence $\lambda_1 \to 0$, contradicting (ii). □

Remark III.2 (What the chain says)**.** *Theorem III.1 establishes: in the no-blow-up branch, each computable NS flow is computationally bounded, geometrically distinguishable, statistically accessible, and informationally compressible. None of these flows approach the regime where K-inside-B drives FIM collapse. The flow retains structure that a computable observer can detect at every time.*

IV. DISTINGUISHABILITY AND PROVABILITY

The contrapositive chain (Theorem III.1) establishes that no-blow-up flows are informationally tame. The question is whether this tameness extends to proof-theoretic tameness—whether the regularity of each datum can be *certified* with bounded proof complexity.

A. The Computable Certificate

Definition IV.1 (BKM certificate)**.** *For a computable datum e and time horizon T, define:*

$$V(e,T) \;=\; \int_0^T \|\omega(\cdot, s; e)\|_{L^\infty}\, ds. \tag{6}$$

By BKM (Theorem II.4), the flow from datum e is regular on $[0,T]$ *if and only if* $V(e,T) < \infty$*.*

Proposition IV.2 (Computability of the certificate)**.** *If* ($\star_{\mathrm{ex}}$) *is true, then for each computable datum e and each rational* $T > 0$*:*

(a) $V(e,T) < \infty$ *(by regularity).*

(b) $V(e,T)$ *is computable from* (e,T) *(since the flow is computable in the regularity regime,* $\|\omega(t)\|_{L^\infty}$ *is computable for each t, and the integral is computable over a bounded interval).*

(c) ZFC proves $V(e,T) < \infty$ for each specific (e,T) (compute the value; it is a specific rational bound; ZFC verifies the computation).

Remark IV.3 (Where I′ hides). *Proposition IV.2(c) gives: for each (e,T), ZFC proves regularity on $[0,T]$. Regularity of datum e is the universal statement "for all $T > 0$, $V(e,T) < \infty$," i.e., the flow never blows up. ZFC proving each finite-T instance does not automatically give ZFC proving the universal statement. This is the Π_1 gap: each instance is provable, but the infinitary conjunction might not be.*

In scenario R, ZFC proves the universal statement via a uniform a priori bound (or other global argument). In scenario I′, no such uniform argument exists in ZFC, and the proof lengths for individual instances grow uncomputably fast.

B. The Grönwall–FIM Certificate

The persistent distinguishability of Theorem III.1(ii) gives a stronger certificate than bare BKM.

Proposition IV.4 (FIM-controlled BKM bound). *If $\lambda_1(t;e) > 0$ for all $t \in [0,T]$, then the Grönwall bound* (2) *gives:*

$$V(e,T) \;=\; \int_0^T \|\omega\|_{L^\infty}\, ds \;\le\; \frac{1}{C'} \ln \frac{\lambda_1(0;e)}{\lambda_1(T;e)}. \tag{7}$$

Both $\lambda_1(0;e)$ and $\lambda_1(T;e)$ are computable from (e,T) in the regularity regime.

Proof. Rearrange (2): $\int_0^T \|\omega\|_{L^\infty} \le (1/C') \ln(\lambda_1(0)/\lambda_1(T))$. The FIM at any time is computed from the score function of P_t, which is computed from the velocity field, which is computable for computable initial data and finite time in the regularity regime. □

Remark IV.5 (The distinguishability certificate). *Proposition IV.4 converts persistent distinguishability into a quantitative regularity certificate. At each time T, the FIM spectral gap $\lambda_1(T;e)$ is positive (Theorem III.1(ii)) and computable. Its positivity is* witnessed*: a computable estimator achieves finite variance bounded by $1/\lambda_1(T;e)$ (Cramér–Rao). The regularity certificate $V(e,T)$ is bounded by a computable function of this witness.*

This is the content of "distinguishability gives you a computable something": $\lambda_1 > 0$ means a computable statistical test separates the flow from its neighbors, and this separation quantitatively bounds the BKM integral that certifies regularity.

C. The Formal Bridge

To exclude I′, we need: for each datum e, ZFC proves global regularity—not just regularity on each $[0,T]$, but the universal statement "for all T". This requires proof lengths bounded by a computable function of e.

Definition IV.6 (Computable proof certificates). *We say that NS regularity has* computable proof certificates *if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for each computable globally regular datum e, there exists a ZFC-proof of "datum e is globally regular" with Gödel number $\le f(e)$.*

Theorem IV.7 (Conditional I′ exclusion). *Assume:*

(H1) $(\star_{\rm ex})$ is true.

(H2) NS regularity has computable proof certificates (Definition IV.6).

Then $\mathrm{ZFC} \vdash (\star_{\rm ex})$*, and scenario I′ is excluded.*

Proof. Under (H2), let f be the computable bounding function. Define the search procedure $\Pi(e)$: enumerate ZFC-proofs with Gödel number $\le f(e)$; check each for being a valid proof of "datum e is globally regular"; output the first such proof.

Under (H1), every datum is regular, so by (H2), $\Pi(e)$ terminates for every e. Since the search space is bounded by $f(e)$ and proof-checking is decidable, Π is a total computable function.

Crucially, Π is provably total in ZFC: the bound $f(e)$ is computable (hence ZFC-definable), the search is bounded, and (H2) guarantees termination—which ZFC can verify by running $\Pi(e)$ for each e and checking the output is a valid proof.

ZFC can now formalize: "for each e, $\Pi(e)$ outputs a valid proof of regularity of datum e." Since ZFC proves the soundness of its own proof-checking for Σ_1 statements, ZFC derives: for each e, datum e is regular. That is, ZFC $\vdash (\star_{\rm ex})$. □

Remark IV.8 (The role of the bound)**.** *The key step is that f is computable, making the search $\Pi(e)$ provably terminating. Without a computable bound, the search "enumerate all ZFC-proofs until one proves regularity of datum e" is total (under (H1), each instance is provable) but not provably total—ZFC cannot verify that the search terminates without knowing a bound. This is precisely the I' scenario: proofs exist for each instance, but their lengths grow faster than any computable function, making the search procedure non-provably-total.*

Remark IV.9 (Subtlety in the soundness step)**.** *The final step—from "ZFC proves regularity instance by instance via Π" to "ZFC proves the universal statement"—uses the uniform definability of Π. ZFC proves: (1) Π is total (by the computable bound); (2) for each e, $\Pi(e)$ is a valid proof of $R(e)$; (3) ZFC is sound for Σ_1 sentences (provable in ZFC by Σ_1-completeness). Combining: ZFC proves $\forall e\, R(e)$, which is $(\star_{\text{ex}})$.*

This argument is standard in proof theory; the non-trivial content is (H2)—the existence of the computable bound.

V. WHY THE CONDITION SHOULD HOLD

Theorem IV.7 reduces the exclusion of I′ to hypothesis (H2): the existence of computable proof certificates. We now argue that the C2 framework makes (H2) the structurally expected state of affairs.

A. C2 Identifies the Unique Mechanism

The C2 equivalence (1) establishes that for exact NS, three conditions are *the same question*: Does blow-up exist? Does the PDE support unlimited computation? Is the regularity decision problem undecidable?

The equivalence is not an analogy. It is a biconditional: the presence of blow-up *is* the presence of unlimited computation *is* the presence of undecidability. The mechanism that generates proof-theoretic difficulty for $(\star_{\text{ex}})$ is *unlimited computation*, and unlimited computation alone.

In the no-blow-up branch, C2 bounds this mechanism. The energy barrier (Proposition II.2) limits each datum to $B(e)$ simulation steps. The flow has finite computational depth.

B. Informational Tameness Bounds Content

Theorem III.1 extends the computational bound to an informational bound. In the no-blow-up branch:

- The flow retains distinguishability ($\lambda_1 > 0$), so a computable estimator can track it (Cramér–Rao).
- The velocity distribution has bounded Kolmogorov complexity (K-inside-B contrapositive), so it admits a short description at every time.
- The BKM certificate is computable and bounded by the FIM (Proposition IV.4).

The "content" of each regularity instance—the information needed to verify that a specific flow stays smooth—is bounded by computable functions of the datum. A proof of regularity for datum e must certify that $V(e,T) < \infty$ for all T. The certificate at each T has complexity bounded by the flow's informational content at time T, which is itself bounded (Theorem III.1).

For (H2) to fail, the proof complexity of assembling these bounded-content certificates into a global regularity proof must grow uncomputably fast in e, even though:

(a) The flow has bounded computational depth.

(b) Each finite-time certificate is computably bounded.

(c) The flow's distribution is compressible (bounded KC) at every time.

(d) The flow is distinguishable (computable estimator exists) at every time.

C. The Structural Argument Against I′

For I′ to hold, the following must all be true simultaneously:

(I′-1) $(\star_{\text{ex}})$ is true: no blow-up.

(I′-2) ZFC proves regularity for each individual datum (each instance is decidable).

(I′-3) Proof lengths grow faster than any computable function of e.

(I′-4) The source of this growth is *not* unbounded computation (C2 has shown computation is bounded to $B(e)$ steps).

(I′-5) The source is purely proof-theoretic: ZFC lacks an axiom or induction principle needed for a uniform argument.

(I′-6) All models of ZFC + $\neg(\star_{\text{ex}})$ are ill-founded (Shoenfield, since $\neg(\star_{\text{ex}})$ is a false Σ^1_1 statement in this scenario).

Condition (I′-4) is the crux. The C2 equivalence has accounted for *all* the logical structure of $(\star_{\text{ex}})$ through a single mechanism: unlimited computation via blow-up amplification. I′ requires a second, independent source of proof-theoretic complexity—one that makes the analytical verification of regularity uncomputably hard, even though the fluid is not computing anything and retains computable, compressible, distinguishable structure at every time.

Remark V.1 (Comparison with known I′-type phenomena)**.** *The canonical example of a true-but-unprovable* Π_1 *statement is Goodstein's theorem (unprovable in PA, provable in PA* $+\varepsilon_0$*-induction). In that case, the proof-theoretic difficulty has an identified mechanism: transfinite induction up to* ε_0*, which PA lacks.*

For NS regularity in the no-blow-up branch, one would need an analogous mechanism: some specific principle that ZFC lacks but that is required to certify regularity uniformly. The C2 equivalence tells us the only *such principle for* $(\star_{\text{ex}})$ *is computational universality—the ability to encode the halting problem. In the no-blow-up branch, this principle is absent. No substitute mechanism is known or structurally motivated.*

Harvey Friedman's combinatorial independence results from ZFC involve statements whose proof-theoretic strength genuinely exceeds ZFC's. For a natural Π^1_1 *PDE statement to fall into this category would require the PDE's analytical structure to encode ordinals or combinatorial principles beyond ZFC, in a way that has nothing to do with Turing computation. No example of this phenomenon is known.*

D. Ill-Foundedness as Structural Diagnosis

The Shoenfield constraint (I′-6) provides a model-theoretic diagnosis. In I′, every model of ZFC + $\neg(\star_{\text{ex}})$ has an ill-founded membership relation: its "natural numbers" include nonstandard elements, and its "blow-up witness" is a phantom existing only in ill-founded models.

The ill-foundedness of the counter-models mirrors the structural baselessness of the scenario. I′ posits an effect (unprovability) after C2 has identified and removed the cause (unlimited computation). The counter-models are ill-founded because they must conjure a blow-up witness that does not exist in any well-founded model—because no blow-up exists. The scenario is, in a precise model-theoretic sense, without foundation.

VI. CONSEQUENCES OF THE HYPOTHESES

Theorem VI.1 (Conditional trichotomy—weak form)**.** *Under Conjecture I.1 (equivalently, under hypothesis (H2) of Theorem IV.7), the four-way dichotomy reduces to three scenarios:*

(a) ***(R)*** $(\star_{\text{ex}})$ *true and ZFC-provable. The energy barrier caps computation. Regularity is provable by certified distinguishability.*

(b) ***(F)*** $(\star_{\text{ex}})$ *false and ZFC-provable. Blow-up exists from computable data. Exact NS is computationally universal. Full undecidability of the decision problem.*

(c) ***(I)*** $(\star_{\text{ex}})$ *false but ZFC-unprovable. Blow-up exists but the witness is* Δ^2_1*-definable, not ZFC-certifiable. Exact NS is computationally universal. Full undecidability of the decision problem.*

Proof. Theorem IV.7 excludes I′: if $(\star_{\text{ex}})$ is true and (H2) holds, then ZFC proves $(\star_{\text{ex}})$, forcing scenario R. □

Theorem VI.2 (Conditional dichotomy—strong form)**.** *Under Conjecture I.2 (equivalently, under hypothesis (H2) together with blow-up stability under* H^1 *perturbation), the four-way dichotomy reduces to two scenarios:*

(a) ***(R)*** *$(\star_{\text{ex}})$ true and ZFC-provable.*

(b) ***(F)*** *$(\star_{\text{ex}})$ false and ZFC-provable.*

In either case, ZFC decides $(\star_{\text{ex}})$. The Clay problem is ZFC-decidable.

Proof. The weak form excludes I′. For scenario I: if blow-up exists, blow-up stability gives computable blow-up data. A computable datum blowing up in finite time is a Σ_1 statement, verifiable by ZFC (run the computation and observe divergence). So ZFC proves $\neg(\star_{\text{ex}})$, upgrading scenario I to scenario F. □

Remark VI.3 (What is proven, what is not)**.** *The unconditional content is:*

(i) C2 identifies unlimited computation as the unique mechanism generating logical difficulty for $(\star_{\text{ex}})$ (proven).

(ii) In the no-blow-up branch, unlimited computation is absent: the energy barrier caps each datum at $B(e)$ steps (proven).

(iii) The flow is informationally tame: computationally bounded, distinguishable, compressible (Theorem III.1, proven).

(iv) Under $(\star_{\text{ex}})$, $\lambda_1(t;e) \geq \delta(e) > 0$ uniformly for all t (Theorem VII.1, proven conditional on $(\star_{\text{ex}})$).

(v) The infinite-time certification reduces to a finite-interval problem with unconditionally computable length bound (proven).

(vi) The counter-models for I′ are all ill-founded (proven, via Shoenfield).

What is not *proven is the C2 provability hypothesis in either form. The weak form requires proving $(\star_{\text{ex}})$—the Clay problem. The strong form additionally requires blow-up stability—an open PDE problem. These are not failures of the framework but consequences of its structure: the C2 equivalence ensures that every question about the proof-theoretic status of $(\star_{\text{ex}})$ reduces to the Clay question itself (Section VIII).*

VII. THE GAP: AN HONEST ASSESSMENT

The formal gap between Theorem III.1 (informational tameness) and hypothesis (H2) (computable proof certificates) is:

Does bounded informational content of a PDE flow imply bounded proof complexity of its regularity?

This is not a PDE question or a computability question. It is a proof-complexity question about the relationship between the descriptive complexity of a mathematical object and the length of proofs about that object.

A. Three Sufficient Conditions

The gap would close under any of: (a) an a priori computable bound on $V(e,T)$ uniform in T; (b) eventual entry into the perturbative small-data regime with computable entry time; (c) a uniform lower bound $\lambda_1(t;e) \geq \delta(e) > 0$ for all t.

We now show that under $(\star_{\text{ex}})$, all three hold simultaneously: the BKM integral converges, the eventual regularity theorem gives a computable entry time, and the Grönwall bound combined with the scale invariance of λ_1 produces the uniform FIM lower bound.

B. Resolution: BKM Integral Convergence and Scale Invariance

The gap between pointwise and uniform distinguishability closes under $(\star_{\text{ex}})$ via a classical PDE argument combined with the scale invariance of λ_1.

Theorem VII.1 (Uniform distinguishability)**.** *If* $(\star_{\rm ex})$ *is true, then for each computable datum* e*:*

$$W(e) \;:=\; \int_0^\infty \|\omega(\cdot,s;e)\|_{L^\infty}\, ds \;<\; \infty, \tag{8}$$

and consequently

$$\lambda_1(t;e) \;\geq\; \delta(e) \;:=\; \lambda_1(0;e)\,\exp(-C'\,W(e)) \;>\; 0 \quad \textit{for all } t \geq 0. \tag{9}$$

Proof. The argument has two steps: BKM integral convergence and the Grönwall consequence.

Step 1: Eventual regularity and tail bound. The energy inequality gives $2\nu\int_0^\infty \|\nabla u\|_{L^2}^2\, ds \leq \|u_0\|_{L^2}^2$. By pigeonhole, for any $T>0$, there exists $t^* \in [0,T]$ with $\|\nabla u(t^*)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2/(2\nu T)$. Since $\|u(t^*)\|_{L^2} \leq \|u_0\|_{L^2}$ (energy inequality), interpolation gives $\|u(t^*)\|_{H^{1/2}} \leq \|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}$. Choosing T large enough (depending computably on $\|u_0\|_{L^2}$ and $\varepsilon_{\rm FK}$), we obtain $\|u(t^*)\|_{H^{1/2}} < \varepsilon_{\rm FK}$, the Fujita–Kato threshold [7].

Set $T_0(e) := t^*$. The bound $T_0(e) \leq C\left\|u_0^{(e)}\right\|_{L^2}^{\alpha}/(\nu^\beta\,\varepsilon_{\rm FK}^\gamma)$ is a computable function of e, and does *not* require assuming regularity—it uses only the energy inequality, which holds for Leray–Hopf [8] weak solutions unconditionally.

After T_0, the Fujita–Kato perturbative theorem [7] gives explicit decay: $\|\omega(t)\|_{L^\infty} \leq C_0\, t^{-5/4}$ for $t \geq T_0$. Therefore

$$\int_{T_0}^\infty \|\omega\|_{L^\infty}\, ds \;\leq\; C_0\int_{T_0}^\infty s^{-5/4}\, ds \;=\; 4C_0\,T_0^{-1/4} \;<\; \infty. \tag{10}$$

Step 2: Finite-interval bound. Under $(\star_{\rm ex})$, the flow is smooth on $[0, T_0(e)]$ and $T_0(e) < \infty$. Therefore $\|\omega(\cdot,s)\|_{L^\infty} < \infty$ for each $s \in [0,T_0]$, and since T_0 is finite:

$$\int_0^{T_0} \|\omega\|_{L^\infty}\, ds \;<\; \infty. \tag{11}$$

Step 3: Total convergence and Grönwall. Combining (10) and (11): $W(e) = \int_0^\infty \|\omega\|_{L^\infty} < \infty$. The Grönwall bound (2) then gives $\lambda_1(t;e) \geq \lambda_1(0;e)\exp(-C'W(e)) =: \delta(e) > 0$ for all $t \geq 0$. □

Remark VII.2 (Scale invariance is essential)**.** *The uniform bound* (9) *exploits a key structural feature:* λ_1 *is scale-invariant. It is computed from the normalized distribution* $P_t = |u|^2/\|u\|_{L^2}^2$*, which divides out the amplitude. As the flow dissipates energy (*$\|u\| \to 0$*), the FIM does not see the dropping amplitude—it tracks only the* shape *of the energy distribution. The shape's evolution is controlled by vorticity via the FIM evolution equation, and the total vorticity budget* $W(e)$ *is finite. Therefore the exponential factor in the Grönwall bound has a finite asymptote: it stabilizes rather than decaying to zero. Without scale invariance, the energy decay would drag* λ_1 *down regardless of vorticity; with it, only vorticity matters, and vorticity's integrated effect is bounded.*

Corollary VII.3 (Uniform BKM certificate)**.** *Under* $(\star_{\rm ex})$*, the Grönwall-BKM certificate is uniform in* T*:*

$$V(e,T) \;=\; \int_0^T \|\omega\|_{L^\infty}\, ds \;\leq\; W(e) \;\leq\; \frac{1}{C'}\ln\frac{\lambda_1(0;e)}{\delta(e)} \;<\; \infty \quad \textit{for all } T. \tag{12}$$

A single finite number $W(e)$ *certifies global regularity for datum* e*.*

C. The Proof Strategy and Its Circularity

Theorem VII.1 resolves the pointwise/uniform gap: under $(\star_{\rm ex})$, uniform distinguishability holds. Combined with the uniform BKM certificate (Corollary VII.3), this gives a natural proof strategy for each datum e:

1. Compute $W(e) = \int_0^\infty \|\omega\|_{L^\infty}$ (this is computable under regularity: the flow is computable, the integrand is computable, and we have shown the integral converges).

2. Exhibit $W(e)$ as a finite bound.

3. Apply BKM: $W(e) < \infty$ implies global regularity.

The proof of regularity for each datum e reduces to a single computation whose output is a finite number. The proof length is bounded by the computation time of $W(e)$, which is a computable function of e.

Remark VII.4 (The circularity)**.** *The computation of $W(e)$ requires the flow to be globally smooth (otherwise the numerical simulation might not converge). Under $(\star_{\rm ex})$, the flow is smooth and $W(e)$ is computable. But the* totality *of the function $e \mapsto W(e)$—its termination for every e—is precisely $(\star_{\rm ex})$ itself.*

In scenario R, ZFC proves $(\star_{\rm ex})$ by some argument (perhaps the one above, perhaps another), and all certificates are accessible.

In scenario I′, $(\star_{\rm ex})$ is true, so $W(e)$ terminates for every e, and ZFC can verify each instance (by running the computation—Σ_1 completeness). But ZFC cannot prove that the computation terminates for all *e simultaneously, because that requires proving $(\star_{\rm ex})$.*

The circularity is tight: the proof strategy's totality IS the theorem it is trying to prove. I′ would require that this circularity is unbreakable—that no other proof strategy avoids it.

D. Why the Circularity Should Break

The circularity has a structural feature that distinguishes it from genuine independence phenomena.

In I′, ZFC cannot prove the totality of $e \mapsto W(e)$, even though:

(a) $W(e)$ terminates for every e (by truth of $(\star_{\rm ex})$).

(b) The eventual regularity time $T_0(e)$ has a computable a priori bound depending only on $\|u_0\|_{L^2}$—*no regularity assumption needed.*

(c) The tail integral $\int_{T_0}^{\infty} \|\omega\|_{L^\infty}$ has an explicit computable bound from the perturbative decay rate—also no regularity assumption needed for $t \geq T_0$.

(d) The only uncontrolled piece is the head integral $\int_0^{T_0} \|\omega\|_{L^\infty}$, which requires regularity on a *finite* interval of *computably bounded length.*

Points (b) and (c) are unconditional—they hold for Leray–Hopf weak solutions without assuming $(\star_{\rm ex})$. The entire "infinite-time" difficulty has been reduced to a finite-interval problem: certify that $\|\omega\|_{L^\infty}$ stays bounded on $[0, T_0(e)]$, where T_0 is computably controlled.

For I′ to hold, the proof complexity of certifying this finite-interval regularity must grow uncomputably fast in e—not because the fluid is computing unboundedly (C2 bounds it to $B(e)$ steps), not because the interval is long (it is computably bounded), but for some purely proof-theoretic reason with no identified mechanism.

E. The Residual Gap

The honest assessment: the argument above does not formally exclude I′. The circularity—totality of $e \mapsto W(e)$ requires $(\star_{\rm ex})$, which is the target—is real. What the argument achieves is:

(i) Resolution of the pointwise/uniform distinguishability question: under $(\star_{\rm ex})$, $\lambda_1(t; e) \geq \delta(e) > 0$ uniformly (Theorem VII.1).

(ii) Reduction of the infinite-time certification to a finite-interval problem with computably bounded length.

(iii) Identification of the circularity as self-referential: the obstruction to proving $(\star_{\rm ex})$ is exactly $(\star_{\rm ex})$ itself, with no independent proof-theoretic content.

(iv) Elimination of all non-circular sources of difficulty: the tail is controlled unconditionally, the interval length is controlled unconditionally, the flow is not computing unboundedly (C2), the distribution is compressible (K-inside-B contrapositive), and the flow is distinguishable (FIM).

The remaining gap is whether "self-referential circularity with no independent content" can sustain genuine independence from ZFC. For known independence phenomena (Goodstein, Friedman, large cardinals), the independence always has independent content: a specific principle (transfinite induction, combinatorial strength, inaccessibility) that the formal system lacks. I′ would be independence without such a principle—unprovability with no reason.

But we must be honest about why this gap cannot be closed here.

VIII. WHY THE GAP IS THE CLAY PROBLEM

The C2 equivalence (1) identifies three questions as the same:

1. Does blow-up exist?

2. Does exact NS support unlimited computation?

3. Is exact NS regularity undecidable?

This note has revealed a fourth face of the same question:

4. Can I′ be excluded?

The exclusion of I′ requires proving that the function $e \mapsto W(e)$ has a computable bound—equivalently, that ZFC can uniformly certify regularity. But this certification IS ($\star_{\rm ex}$): proving that every datum is regular is the Clay problem. The inability to exclude I′ without solving the Clay problem is not a defect of our analysis. It is a *theorem about the structure of the dichotomy*: I′ is the scenario in which the Clay problem is true but its truth outruns ZFC's deductive capacity, and determining whether this scenario holds requires the same information as solving the Clay problem itself.

In scenarios F and I, the Clay problem is resolved (blow-up exists), and I′ is moot—it applies only to the ($\star_{\rm ex}$)-true branch.

In scenario R, the Clay problem is resolved (regularity is proved), and I′ is excluded by definition—R means ZFC proves it.

I′ is precisely the scenario where the Clay problem has an answer (regularity holds) but no proof. Determining whether such a scenario obtains requires knowing whether the answer exists, which requires knowing the answer. The circularity is not an artifact of our proof strategy; it is the logical structure of the problem.

Remark VIII.1 (Comparison with Remark 8.5 of [1])**.** *The main paper's Remark 8.5 observes that the C2 equivalence makes "does blow-up exist?" ⇔ "does exact NS compute?" ⇔ "is regularity undecidable?" and notes: "the circularity is the content of the theorem." The present note extends this observation: the question "is I′ the correct scenario?" is also equivalent to the Clay question, via the same C2 mechanism. Every attempt to resolve the dichotomy from within the framework returns to the same boundary. The framework does not solve the Clay problem. It characterizes what each possible solution means and shows that no resolution of the dichotomy can be achieved independently of the Clay question.*

IX. CONCLUSION

This note proves three things and characterizes a fourth.

Proven:

(i) The contrapositive chain: in the no-blow-up branch, each flow has bounded computational depth, persistent distinguishability, compressible distributions, and computable BKM certificates (Theorem III.1).

(ii) Uniform distinguishability: under ($\star_{\rm ex}$), $\lambda_1(t;e) \geq \delta(e) > 0$ for all t, by scale invariance of the FIM and the finite vorticity budget (Theorem VII.1).

(iii) Conditional I′ exclusion: if regularity has computable proof certificates, ZFC proves ($\star_{\rm ex}$) and I′ is excluded (Theorem IV.7).

Characterized but not proven:

(iv) The C2 provability hypothesis, in two forms:

- **Weak** (Conjecture I.1): no blow-up ⇒ ZFC proves ($\star_{\rm ex}$). Eliminates I′. Surviving scenarios: R, F, I. Gap: proving this requires proving ($\star_{\rm ex}$) (the Clay problem).
- **Strong** (Conjecture I.2): ZFC decides ($\star_{\rm ex}$) one way or the other. Eliminates both I′ and I. Surviving scenarios: R or F. Additional gap: blow-up stability under H^1 perturbation (a PDE assumption).

The C2 equivalence compresses the entire landscape of exact NS regularity into a single binary question: does the physical nonlinearity support unlimited Turing-complete computation? This note shows that the proof-theoretic status of $(\star_{\mathrm{ex}})$—whether it is provable, independent, or refutable in ZFC—is also determined by the answer to this question. Each scenario in the dichotomy tells a different computational story about the Clay problem.

In **scenario R**, the fluid does not blow up. The energy barrier caps every datum at finitely many simulation steps. No computable initial datum can encode a non-halting Turing machine indefinitely—viscosity destroys the encoding before it runs forever. The Church–Turing barrier does not engage, because the system is not computationally universal. Regularity is provable: ZFC can certify, datum by datum, that the fluid's bounded computational capacity never produces a singularity.

In **scenario F**, the fluid blows up, and ZFC can prove it. Blow-up amplification—the growing amplitude near a singularity—outpaces viscous dissipation, enabling a computable initial datum to simulate a non-halting Turing machine forever. The system becomes computationally universal. The Church–Turing barrier engages in full: deciding which computable data are regular is equivalent to deciding which Turing machines halt. Some data provably blow up (encoding non-halting machines), others are provably regular (encoding halting machines), and the universal regularity statement is false and PA-refutable.

In **scenario I**, the fluid blows up, but ZFC cannot prove it. The blow-up datum exists—it is Δ^2_1-definable by Shoenfield—but ZFC lacks the deductive strength to witness it. The system is computationally universal (as in F), the Church–Turing barrier engages (as in F), but the blow-up itself sits beyond ZFC's reach. The truth value is the same in every transitive model (Shoenfield absoluteness), so the independence is a genuine gap in ZFC's power, not a set-theoretic ambiguity.

In **scenario I′**, the fluid does not blow up, but ZFC cannot prove it. The energy barrier caps computation (as in R), the flow is distinguishable and compressible (as in R), and no Church–Turing obstruction applies (as in R). Yet ZFC cannot assemble the individual regularity certificates into a uniform proof. The difficulty is not computational—C2 has established that—but purely proof-theoretic, arising from a source unrelated to Turing computation. All counter-models of $(\star_{\mathrm{ex}})$ are ill-founded. No precedent exists for this phenomenon among natural Π^1_1 PDE statements.

Which scenario holds is the Clay problem. The C2 equivalence does not answer this question. It reveals that the question has exactly four possible answers, that each answer has precise computational content, and that no resolution of the dichotomy can be achieved independently of the question itself.

Under the weak hypothesis (Conjecture I.1), three scenarios survive: R, F, or I. Under the strong hypothesis (Conjecture I.2), two survive: R or F—the Clay problem is ZFC-decidable. Whether one of these holds, and which one, is not a question this framework can answer. It is the question the framework was built to ask.

[1] D. Christian, "Navier–Stokes regularity is independent of ZFC: the Church–Turing barrier and the Fisher information spectral gap," 2026. Primary paper.

[2] D. Christian, "Navier–Stokes regularity and the Fisher information spectral gap: APO foundations and Bhattacharyya–Fisher identity," 2026. Companion paper.

[3] J. R. Shoenfield, "The problem of predicativity," *Essays on the Foundations of Mathematics*, 132–139, 1961.

[4] M. Li and P. Vitányi, *An Introduction to Kolmogorov Complexity and Its Applications*, 3rd ed., Springer, 2008.

[5] J. T. Beale, T. Kato, and A. Majda, "Remarks on the breakdown of smooth solutions for the 3-D Euler equations," *Commun. Math. Phys.* **94**, 61–66 (1984).

[6] E. L. Lehmann and G. Casella, *Theory of Point Estimation*, 2nd ed., Springer, 1998.

[7] H. Fujita and T. Kato, "On the Navier–Stokes initial value problem. I," *Arch. Rational Mech. Anal.* **16**, 269–315 (1964).

[8] J. Leray, "Sur le mouvement d'un liquide visqueux emplissant l'espace," *Acta Math.* **63**, 193–248 (1934).